Sampling Conditions for Conforming Voronoi Meshing by the VoroCrust Algorithm

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Abstract

We study the problem of decomposing a volume with a smooth boundary into a collection of Voronoi cells. Unlike the dual problem of conforming Delaunay meshing, a principled solution to this problem for generic smooth surfaces remained elusive. VoroCrust leverages ideas from weighted $\alpha$-shapes and the power crust algorithm to produce unweighted Voronoi cells conforming to the surface, yielding the first provably-correct algorithm for this problem. Given a $\kappa$-sparse $\epsilon$-sample, we work with the balls of radius $\delta$ times the local feature size centered at each sample. The corners of the union of these balls on both sides of the surface are the Voronoi sites and the interface of their cells is a watertight surface reconstruction embedded in the dual shape of the union of balls. With the surface protected, the enclosed volume can be further decomposed by generating more sites inside it. Compared to clipping-based algorithms, VoroCrust cells are full Voronoi cells, with convexity and fatness guarantees. Compared to the power crust algorithm, VoroCrust cells are not filtered, are unweighted, and offer greater flexibility in meshing the enclosed volume by either structured or randomly generated samples.

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1 Introduction

Generating high-quality meshes is an important problem in computational geometry, computer graphics and geometric modeling. In the simulation community, there has been a growing interest in polyhedral meshes because of its advantages over tetrahedral or hex-dominant meshes: for example, higher degrees of freedom per element, and producing fewer elements for the same number of vertices. Voronoi cells share several geometric properties with tetrahedra, e.g., planar facets and positive Jacobians, making them particularly suitable for numerical simulations.

VoroCrust is the first provably-correct solution to generating a 3D Voronoi mesh with cells of bounded aspect ratio, whose boundary also conforms to a smooth 2D surface. A conforming volume mesh exhibits two desirable properties simultaneously: 1) a decomposition of the enclosed volume, and 2) a reconstruction of the bounding surface. Meshing by Delaunay tetrahedra is well-studied [17]. For Voronoi meshing, a common technique to produce boundary-conforming cells relies on clipping, i.e., intersecting and truncating, each
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cell by the bounding surface [25]. An alternative to clipping is to locally mirror the Voronoi
generators on either side of the surface [10].

VoroCrust can be viewed as a principled mirroring technique. Similar to the power
crust [6], the reconstruction is composed of the facets shared by cells on the inside and
outside of the manifold. However, VoroCrust uses pairs of unweighted generators tightly
hugging the surface, which allows further decomposition of the interior without disrupting
the surface reconstruction. VoroCrust is closely related to a special class of weighted α-
shapes [26], defined within a general framework for non-uniform approximation [16]. See
Appendix A for additional information on VoroCrust’s relationship to the literature, and
some hints of VoroCrust’s practicality and utility. A description of the abstract VoroCrust
algorithm we analyze is given next. Figure 1 illustrates the steps in 2D.

The Abstract VoroCrust Algorithm

1. Take as input an \( \sigma \)-sparse \( \epsilon \)-sampling \( P \) on the bounding surface \( M \) of the volume \( O \).
2. Define a ball \( B_i \) of radius \( r_i = \delta \text{ls}(p_i) \) centered at each sample \( p_i \). Let \( U = \bigcup_i B_i \).
3. The corner points of \( \partial U \) yield the Voronoi generators \( G = G_I \cup G_O \) inside and outside \( O \).
4. Include in \( G \) more generators, away from \( M \), inside \( O \) to further decompose it.
5. Compute the Voronoi diagram \( \text{Vor}(G) \). The mesh \( \hat{O} = \bigcup_{g \in G_I} \text{Vor}(g) \) approximates \( O \) and
the facets separating \( G_I \) and \( G_O \) form the surface reconstruction \( \hat{M} \) approximating \( M \).

**Problem Statement:** We seek to characterize the locations and weights of the input
samples to guarantee a topologically-correct and geometrically-accurate reconstruction along
with a basic approach for decomposing the interior.

![Figure 1 VoroCrust reconstruction, demonstrated on a planar curve. The weight of a point
defines the radius of a ball around it. The reconstruction is the Voronoi facets separating the
uncovered intersection pairs on opposite sides of the manifold.](image)

Summary of Contributions: For a certain choice of \( \epsilon, \delta \) and \( \sigma \), we show that the
reconstruction \( \hat{O} \) is ambient isotopic to \( O \) and every cell in \( \text{Vor}(G_I) \) is fat with a bounded
circumradius-to-inradius ratio. The number of samples can be bounded by an integral related
to the local feature size function over \( O \), meaning it is optimal up to a constant factor.

2 Definitions and Preliminaries

The Voronoi diagram of a set of seed points is the cell complex induced by the partition
of space into the regions closer to each seed than any other. The power diagram, or Laguerre
diagram, is defined similarly using the power distance instead of the standard Euclidean
distance; it is a weighted Voronoi diagram. Let \( \text{Vor}(P) \) and \( \text{Del}(P) \) denote the Voronoi
diagram of the set \( P \) and its dual Delaunay triangulation. Similarly, let \( \text{wVor}(P) \) and
\( \text{wDel}(P) \) be their weighted versions.

**Definition 1 (Power Distance).** The weighted distance or \textit{power} of \( p_i \) at \( x \in \mathbb{R}^3 \) is \( \pi_{p_i}(x) = ||p_i x||^2 - w_i \), where \( w_i = r_i^2 \) is the weight of the sample \( p_i \).
Note that the bisector, i.e., equidistant separator or chordale, between two samples is a plane, just as with Euclidean distance. There is a natural geometric interpretation of the weight \( w_i \). If we draw the sphere \( S_i \) with radius \( r_i = \sqrt{w_i} \) at \( p_i \), then points on \( S_i \) have \( \pi_{p_i}(.) = 0 \), inside \( \pi_{p_i}(.) < 0 \) and outside \( \pi_{p_i}(.) > 0 \).

A tetrahedron in \( \text{wDel} \) is dual to a power vertex, its weighted circumcenter, whose weighted distances to each of the tetrahedron’s vertices are equal and called the weighted circumradius. For tetrahedra with negative circumradius, the power vertex lies inside each sample’s sphere. Similarly, each triangle has a weighted circumcenter whose weighted distance to any of the triangle vertices is the weighted circumradius of the triangle.

2.1 The Dual Shape of \( U \)

Central to the method and analysis are the guide points arising from triples of sample spheres that intersect at two points contributing a guide triangle to the \( \text{wDel} \). The reconstruction consists of Voronoi facets, most of which are guide triangles.

- **Definition 2** (Guides). If spheres \( \{S_i, S_j, S_k\} \) intersect at exactly two points, the intersection points \( g^\uparrow_{ijk} = \{g^\uparrow_{ijk}, g^\downarrow_{ijk}\} \) are a pair of guide points or guides. We say \( \triangle_{ijk} \) is dual to \( g^\uparrow_{ijk} \) and use arrows to distinguish guides on different sides of the manifold with \( g^\uparrow \) being interior and \( g^\downarrow \) exterior.

- **Definition 3** (Seeds and Covered Guides). A guide point \( g \) which is not interior to any sample sphere is uncovered and used as a seed \( G \) for the Voronoi diagram; covered guides are not. We use capitalization to distinguish uncovered guides \( G \) from covered guides \( g \), whenever coverage is known and important. If one guide of a pair is covered, then we say the guide pair is half-covered. If both guides in a pair are covered, they are ignored. Let \( \mathcal{G} \) denote the set of all seeds and \( \mathcal{G}_i = \mathcal{G} \setminus S_i \).

- **Definition 4** (Guide Triangles). \( \triangle_{ijk} \) is called a type-1 guide triangle if it generates exactly one seed and type-2 guide triangle if it generates two seeds. Denote the set of all guide triangles by \( \mathcal{T}_G = \mathcal{T}_G(\mathcal{P}) \).

- **Definition 5** (Guide Edges and Circles). \( e_{ij} = \overline{p_i p_j} \) is a guide edge if it appears in some guide triangle \( \triangle_{ijk} \), and has an associated guide circle \( C_{ij} = S_i \cap S_j \).

Since pairs of guides which are both covered are not used as Voronoi seeds, they are irrelevant to our analysis. Other configurations of overlapping spheres that do not produce guides are also irrelevant. For ease of exposition, we assume general position for both samples and weights, e.g., sample spheres are not tangent and no four intersect at exactly two points.

While we make frequent reference to guide triangles and their faces, it is important to recognize the global structure of the complex \( \mathcal{K} = \text{Nerve}(\{\text{Vor}(p_i) \cap B_i \mid p_i \in \mathcal{P}\}) \). Recall that the nerve theorem implies \( \mathcal{S} = |\mathcal{K}| \) has the same homotopy-type as \( U \), where \( \mathcal{S} \) is the underlying space of \( \mathcal{K} \). As established in [26], the 2-simplices of \( \mathcal{K} \) corresponding to a subset of our guide triangles can be classified depending on the number of points on \( \partial U \) that witness the existence of the simplex in the complex, where the witness points are the Voronoi seeds chosen by the VoroCrust algorithm. It is easy to see that for guide triangles with two such witness points are immediately recovered as facet shared among the Voronoi cells of those two points. This is because the two witnesses lie on the dual face of the triangle and it follows that all points on the facet are equidistant to both witnesses. For the other case with a single witness, the Voronoi cells of the witnesses include the triangle and extend beyond it yielding an extra Voronoi vertex as a Steiner point in the output mesh;
see Figure 3. This intuition will be made explicit in Section 3 where we establish that the VoroCrust is embedded in $\mathcal{S}$.

2.2 Reconstruction Basics

A crucial property for the correctness of our reconstruction concerns how sample balls $B_i$ overlap and many of our proofs examine such intersection patterns on their surfaces defined by the sample spheres $S_i = \partial B_i$, see Figure 2.

Definition 6 (Caps and Bands). For $S_i$, let $K^\uparrow_i$ and $K^\downarrow_i$ be the subset of $S_i$ outside all other spheres that lie in the interior and exterior of $\mathcal{M}$, respectively. The subset of $S_i$ inside other spheres is called the medial band.

Observe that all upper seeds $G^\uparrow_i$ lie on $\partial K^\uparrow_i$, formed by the arcs arising from the intersections of other spheres with $S_i$. Similarly, all lower seeds $G^\downarrow_i$ lie on $\partial K^\downarrow_i$.

Definition 7 (Disk Caps). The sample sphere $S_i$ has disk caps if it is partitioned by the other spheres into a covered medial band separating an uncovered upper cap $K^\uparrow_i \neq \emptyset$ and an uncovered lower cap $K^\downarrow_i \neq \emptyset$, with the two caps being topological disks and the medial band a topological annulus.

The importance of disk caps is made clear by the following observation, and in Section 4, we derive sufficient conditions on sampling to ensure this property holds.

Observation 8 (Three upper/lower seeds). If $S_i$ has disk caps, then each of $\partial K^\uparrow_i$ and $\partial K^\downarrow_i$ has at least three seeds and the seeds on $S_i$ are not all co-planar.

Proof. For any $S_j, j \neq i$, such that $S_i \cap S_j \neq \emptyset$, $C_{ij}$ covers strictly less than one hemisphere of $S_i$. Otherwise either $K^\uparrow_i = \emptyset$ or $K^\downarrow_i = \emptyset$, while $S_i$ has disk caps by assumption. Hence, each cap is composed of at least three arcs connecting at least three upper seeds $G^\uparrow_i \subset \partial K^\uparrow_i$ and three lower seeds $G^\downarrow_i \subset \partial K^\downarrow_i$. As $G^\uparrow_i \subset S_i$, they cannot be collinear, and similarly for $G^\downarrow_i$. It follows that the set of seeds $G_i = G^\uparrow_i \cup G^\downarrow_i$ is not coplanar.

Figure 2 Caps and medial band.
**Corollary 9 (Sample Reconstruction).** If \( S_i \) has disk caps, then \( p_i \) is reconstructed as a Voronoi vertex.

**Proof.** By Observation 8, the sample is equidistant to at least four seeds which are not all coplanar. It follows that the sample appears as a vertex in the Voronoi diagram and not in the relative interior of a facet or an edge. Being a common vertex to at least one interior and one exterior Voronoi seeds, VoroCrust retains this vertex in its output reconstruction. ▶

### 3 Sandwiching the Reconstruction in the Dual Shape of \( U \)

Triangulations of curved surfaces embedded in \( \mathbb{R}^3 \) can have half-covered guides pairs, with one guide covered by a sphere centered at a fourth sample which is not a vertex of the associated guide triangle. The tetrahedron formed by the three samples of the guide triangle plus the fourth covering sample is a *sliver*. In this case we do not reconstruct the guide triangle, and also do not reconstruct some guide edges. The goal of this section is to prove, through a series of lemmas, that the reconstructed surface lies entirely within the region of space bounded by guide triangles as stated in the following theorem. Throughout this section, we assume all sample spheres have disk caps, see Definition 7.

**Theorem 10 (Sphere Sandwich).** Assuming all sample spheres have disk caps, the surface reconstruction lies entirely between guide triangles.

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**Figure 4** Sliver tetrahedron, exaggerated scale. \( G^{\downarrow}_{123} \) and \( G^{\downarrow}_{134} \) are the uncovered lower guide seeds, with \( g^{\downarrow}_{123} \) and \( g^{\downarrow}_{134} \) covered. The uncovered upper guide seeds are \( G^{\uparrow}_{124} \) and \( G^{\uparrow}_{234} \), with \( g^{\uparrow}_{124} \) and \( g^{\uparrow}_{234} \) covered. Facet \( f_{ac} \) separates seeds \( ^{\ast}G^{\downarrow}_{123} \) and \( ^{\ast}G^{\downarrow}_{124} \), etc. Voronoi vertex \( n \) is a Steiner point, and in general is not the sliver circumcenter.

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Figure 4a shows the simple case of a single isolated sliver tetrahedron. A guide triangle is an *upper (lower) guide triangle* if its upper (lower) guide point is uncovered. A sliver has a pair of lower guide triangles and a pair of upper guide triangles. For instance, \( \Delta_{124} \) and \( \Delta_{234} \) are the pair of upper triangles in Figure 4a. There is a sliver edge between any pair of samples corresponding to a non-empty circle of intersection between sample spheres, e.g., \( C_{34} \) in Figure 4b. For this circle, the arcs covered by the other two spheres of the sliver overlap, so each of these spheres provides exactly one uncovered seed, rather than two. In this way the upper guides for the upper triangles are uncovered, but their lower
4 Sampling Conditions and Approximation Guarantees

We take as input a set of points $P$ sampled from the bounding surface $M$ such that $P$ is an $\epsilon$-sample, with $\epsilon \leq 1/500$. We require that $P$ satisfies the following sparsity condition: for any two points $p_i, p_j \in P$, $\text{lfs}(p_i) \geq \text{lfs}(p_j) \implies d(p_i, p_j) \geq \sigma \text{lfs}(p_j)$, with $\sigma \geq 3/4$. For each sample $p_i$, we include the ball $B_i = B(p_i, r_i)$ into $U = \bigcup_i B_i$, with $r_i = \delta \text{lfs}(p_i)$ and $\delta = 2\epsilon$. This is equivalent to the weight assignment $p_i \mapsto w_i = r_i^2$. The corner points of $\partial U$, where three balls intersect, yield two sets of seeds $G_I$ and $G_O$ that lie inside and outside the volume, respectively.

Such a sampling $P$ can be obtained by known algorithms, assuming certain representations of the bounding surface $M$. The algorithm in [11] computes a loose $c'$-samples $E$ which is a $c'(1 + 8.5\epsilon')$-sample, given access to an oracle that determines if a point lies in the enclosed volume and whether a line segment intersects the surface along with such a point of intersection. More specifically, whenever the algorithm inserts a new sample $p$ into the set $E$, $d(p, E) \geq \epsilon' \text{lfs}(p)$. To obtain $E$ as an $\epsilon$-sample, we set $\epsilon'(\epsilon) = (\sqrt{34\epsilon + 1} - 1)/17$. Observing that $3\epsilon/4 \leq \epsilon'(\epsilon)$ for $\epsilon \leq 1/40$, the returned $\epsilon$-sample satisfies the desired sparsity condition with $\sigma \geq 3/4$ as required.

We start by adapting Theorem 6.2 and Lemma 6.4 from [16] to the setting just described. For $x \in \mathbb{R}^3 \setminus M$, let $g(x) = d(x, \bar{x})/\text{lfs}(\bar{x})$, where $\bar{x}$ is the closest point to $x$ on $M$. Let $[ab]$ denote the line segment between $a$ and $b$.

▶ Corollary 11. For an $\epsilon$-sample $P$, with $\epsilon \leq 1/20$, the union of balls $U$ with $\delta = 2\epsilon$ satisfies:
1. $M$ is a deformation retract of $U$,
2. $\partial U$ contains two connected components, each isotopic to $M$,
3. $g^{-1}([0, a']) \subset U \subset g^{-1}([0, b'])$, where $a' = \epsilon - 2\epsilon^2$ and $b' \leq 40\epsilon/16$.

Proof. Theorem 6.2 from [16] is stated for balls with radii within $[a, b]$ times the lfs. We set $a = b = \delta$ and use $\epsilon \leq 1/20$ to simplify fractions. This yields the above expressions for $a' = (1 - \epsilon)\delta - \epsilon$ and $b' = \delta/(1 - 2\delta)$. The general condition requires $CL(\epsilon) = (1 - a')^2 + (b' - a')^2 (1 + 2b' - a')/(1 - \delta)^2 < 1$, as we assume no noise. Plugging in the values of $a'$ and $b'$, we verify that the inequality holds for the chosen range of $\epsilon$.

Furthermore, we require that each ball $B_i \subset U$ contributes one facet to each side of $\partial U$. Letting $n_{p_i}$ be the line through $p_i$ perpendicular to $M$, we define the two intersection points $n_{p_i} \cap \partial B_i$ as the poles of $B_i$. Our sampling conditions ensure that both poles are outside any ball $B_j \subset U$.

▶ Lemma 12. All balls in $U$ have uncovered poles for $\epsilon \leq 0.066$, $\delta = 2\epsilon$ and $\sigma \geq 3/2$.

Proof. Fix a sample $p_i$ and let $x$ be one of the poles of $B_i$ and $B_x = B(c, \text{lfs}(p_i))$ the tangent ball at $p_i$ with $x \in B_x$. Letting $p_j$ be the closest sample to $x$ in $P \setminus \{p_i\}$, we assume the worst case where $\text{lfs}(p_j) \geq \text{lfs}(p_i)$ and $p_j$ lies on $\partial B_x$. To simplify the calculations, take $\text{lfs}(p_i) = 1$ and let $\ell$ denote $d(p_i, p_j)$. As lfs is 1-Lipschitz, we get $\text{lfs}(p_j) \leq 1 + \ell$.

By the law of cosines, $d(p_i, x)^2 = d(p_i, p_j)^2 + d(p_i, x)^2 - 2d(p_i, p_j)d(p_i, x)\cos(\phi)$, where $\phi = \angle p_j p_i c$. Letting $\theta = \angle p_i c p_j$, observe that $\cos(\phi) = \sin(\theta/2) = \ell/2$. To enforce $x \notin B_j$, we require $d(p_i, x) > \delta \text{lfs}(p_i)$, which is equivalent to $\ell^2 + \ell^2 - 2\ell^2 > \delta^2(1 + \ell)^2$. Simplifying, we get $\ell > 2\delta^2/(1 - \delta - \delta^2)$. The sparsity of the sampling guarantees $\ell > \sigma \epsilon$. Setting
\[\sigma > 2\delta^2/(1 - \delta - \delta^2)\] we obtain \[4\sigma\epsilon^2 + (8 + 2\sigma)\epsilon - \sigma < 0,\] which requires \(\epsilon < 0.066\) when \(\sigma \geq 3/4.\)

Corollary 11 together with Lemma 12 imply that each \(B_i\) is decomposed into a covered region \(\partial B_i \cap \bigcup_{j \neq i} B_j\), which we call a medial band, and two uncovered regions \(\partial B_i \setminus \bigcup_{j \neq i} B_j\), each containing one pole, which we call caps. Recalling that seeds arise as the pairs of intersection points among the boundaries of such balls, we establish that seeds can be classified correctly as either inside or outside \(M\).

**Corollary 13.** If a seed pair lies on the same side of \(M\), then at least one of them is covered.

**Proof.** Fix such a seed pair \(\partial B_i \cap \partial B_j \cap \partial B_k\) and recall that \(M \cap \partial B_i\) is contained in the medial band on \(\partial B_i\). Now, assume for contradiction that both seeds lie on the same side of \(M\). It follows that \(B_j \cap B_k\) intersects \(B_i\) away from its medial band, a contradiction to Corollary 11.

Corollary 11 guarantees that the medial band of \(B_i\) is a superset of \(g^{-1}([0, a']) \cap \partial B_i\), which means that all seeds \(g_{ijk}\) are at least \(a'\text{lfs}(G_{ijk})\) away from \(M\). It will be useful to bound the elevation of such seeds above \(T_p\), the tangent plane to \(M\) at \(p_i\).

**Lemma 14.** For a seed \(s \in \partial B_i\), \(\theta_x = \angle spRs' \geq 29.34^\circ\) and \(\theta_x > \frac{1}{2} - 5\epsilon\), where \(s'\) is the projection of \(s\) on \(T_p\), implying \(d(s, s') \geq h^+_{\epsilon}\text{lfs}(p_i)\), with \(h^+_{\epsilon} > 0.46\) and \(h^+_{\epsilon} > \frac{1}{2} - 5\epsilon\).

**Proof.** Let \(\text{lfs}(p_i) = 1\) and \(B_i = B(c, 1)\) be the tangent ball at \(p_i\) with \(s \notin B_i\); see Figure 5a. Observe that \(d(s, M) \leq d(s, x)\), where \(x = [sc] \cap \partial B_s\). By the law of cosines, \(d(s, c)^2 = d(p_i, c)^2 + d(p_i, s)^2 - 2d(p_i, c)d(p_i, s)\cos(\pi/2 + \theta_x) = 1 + \delta^2 + 2\delta\sin(\theta_x)\). We may write \(d(s, c) \leq 1 + \delta^2/2 + \delta\sin(\theta_x)\). As \(\text{lfs}\) is \(1\)-Lipschitz and \(d(p_i, x) \leq \delta\), we get \(1 - \delta \leq \text{lfs}(s) \leq 1 + \delta\). There must exist a sample \(p_j\) such that \(d(x, p_j) \leq \text{lfs}(x) \leq \epsilon(1 + \delta)\). Similarly, \(\text{lfs}(p_j) \geq (1 - \epsilon(1 + \delta))(1 - \delta)\). By the triangle inequality, \(d(s, p_j) \leq d(s, x) + d(x, p_j) \leq \delta^2/2 + \delta\sin(\theta_x) + \epsilon(1 + \delta)\). Setting \(d(s, p_j) < \delta(1 - \delta)(1 - \epsilon(1 + \delta))\) implies \(d(s, p_j) < \epsilon\text{lfs}(p_j)\), which shows that for small values of \(\theta_x\), \(s\) cannot be a seed and \(p_j \neq p_i\). Plugging in \(\delta = 2\epsilon\), we get \(\theta_x \geq \sin^{-1}(2\epsilon^3 - 5\epsilon + 1/2) \geq 29.34^\circ\) and \(\theta_x > 1/2 - 5\epsilon\).

**Observation 15.** \(B_i \cap B_j \neq \emptyset \implies d(p_i, p_j) \leq \kappa\delta \cdot \text{lfs}(p_i)\), with \(\kappa = 2/(1 - \delta) < 2.11\) and \(d(p_i, p_j) \geq \kappa \cdot \text{lfs}(p_i)\) with \(\kappa = \sigma\epsilon/(1 + \sigma\epsilon)\).

**Proof.** The upper bound comes from \(d(p_i, p_j) \leq r_i + r_j\) and \(\text{lfs}(p_j) \leq \text{lfs}(p_i) + d(p_i, d_j)\) as \(\text{lfs}\) is \(1\)-Lipschitz, and the lower bound from \(\text{lfs}(p_i) - d(p_i, d_j) \leq \text{lfs}(p_j)\) and the sparsity criteria.

Bounding the circumradii is the culprit behind the small values of \(\epsilon\) we need.

**Lemma 16.** The circumradius of a guide triangle \(t_{ijk}\) is at most \(\varrho_f \cdot \delta\text{lfs}(p_i)\), where \(\varrho_f < 1.38\), and at most \(\varrho_f d(p_i, p_j)\) where \(\varrho_f < 3.68\).

**Proof.** Let \(p_i\) and \(p_j\) be the triangle vertices with the smallest and largest \(\text{lfs}\) values, respectively. From Observation 15, we get \(d(p_i, p_j) \leq \kappa\delta\text{lfs}(p_i)\). It follows that \(\text{lfs}(p_i) \leq (1 + \kappa)\delta\text{lfs}(p_i)\). As \(t_{ijk}\) is a guide triangle, we know that it has a pair of intersection points \(\partial B_i \cap \partial B_j \cap \partial B_k\). Clearly, the seed is no farther than \(\delta\text{lfs}(p_j)\) from any vertex of \(t_{ijk}\) and the orthoradius of \(t_{ijk}\) cannot be bigger than this distance.
Recall that the weight \( w_i \) associated with \( p_i \) is \( \delta^2 lfs(p_i)^2 \). We shift the weights of all the vertices of \( t_{ijk} \) by the lowest weight \( w_i \), which does not change the orthocenter. With that \( w_j - w_i = \delta^2 lfs(p_j)^2 - lfs(p_i)^2 \leq \delta^2 lfs(p_i)^2((1 + \kappa \delta)^2 - 1) = \kappa \delta^2 lfs(p_i)^2(\kappa \delta + 2) \).

On the other hand, sparsity ensures that the closest vertex in \( t_{ijk} \) to \( p_i \) is at distance at least \( N(p_j) \geq \sigma lfs(p_j) \geq \sigma(1 - \kappa \delta) lfs(p_i) \). Ensuring \( \alpha^2 \leq (w_j - w_i)/N(p_j)^2 \leq \kappa \delta^2(2 + \kappa \delta)/(\sigma^2 lfs(p_i)^2(1 - \kappa \delta)) \leq 1/4 \) suffices to bound the circumradius of \( t_{ijk} \) by \( c_{rad} = 1/\sqrt{1 - 4\alpha^2} \) times its orthoradius, as required by the Triangle Radius Lemma (Lemma 10.2 in [17]). Plugging in \( \delta = 2\epsilon \) and \( \sigma \geq 3/4 \) we get \( \alpha^2 \leq 78.97\epsilon \), which corresponds to \( c_{rad} < 1.37 \). It follows that the circumradius is at most \( c_{rad} \delta lfs(p_j) \leq c_{rad}(1 + \kappa \delta) lfs(p_i) < 1.38 \delta lfs(p_i) \).

For the second statement, observe that \( lfs(p_i) \geq (1 - \kappa \delta) lfs(p_j) \) and the sparsity condition ensures that the shortest edge length is at least \( \sigma lfs(p_i) \geq \sigma(1 - \kappa \delta) lfs(p_j) \). It follows that the circumradius is at most \( \frac{\delta c_{rad}}{\sigma(1 - \kappa \delta)} < 3.68 \) times the length of any edge of \( t_{ijk} \).

Given the bound on the circumradii, we are able to bound the deviation of normals.

**Lemma 17.** If \( t_{ijk} \) is a guide triangle, then (1) \( \angle_a(n_{p_i}, n_{p_j}) \leq \eta_0 \delta < 0.47^\circ \), with \( \eta_0 < 2.03 \), and (2) \( \angle_a(n_i, n_p) \leq \eta_\delta < 1.52^\circ \), with \( \eta_\delta < 6.6 \), where \( n_p \) is the line normal to \( M \) at \( p_i \) and \( n_t \) is the normal to \( t_{ijk} \). In particular, \( t_{ijk} \) makes an angle at most \( \eta_\delta \) with \( T_p \).

**Proof.** As \( p_i \) and \( p_j \) are vertices of a guide triangle, Observation 15 implies \( d(p_i, p_j) \leq \kappa \delta lfs(p_i) \) and (1) follows from the Normal Variation Lemma [7] with \( \rho = \kappa \delta < 1/3 \) yielding \( \angle_a(n_{p_i}, n_{p_j}) \leq \kappa \delta/(1 - \kappa \delta) \). Letting \( R_t \) denote the circumradius of \( t \), Lemma 16 implies that the \( R_t \leq g_f \cdot \delta lfs(p_i) \leq lfs(p_i)\sqrt{2} \) and the Triangle Normal Lemma [19] implies \( \angle_a(n_{p_\ast}, n_{t}) \leq 4.57^\circ < 1.05^\circ \), where \( p_\ast \) is the vertex of \( t \) subtending a maximal angle in \( t \). Hence, \( \angle_a(n_{p_i}, n_t) \leq \angle_a(n_{p_\ast}, n_{p_r}) + \angle_a(n_{p_r}, n_t) \).

Towards establishing homeomorphism, the next lemma on the monotonicity of distance to the nearest seed is critical. First, we point out direct consequence of the weighted sampling.

**Observation 18.** \( \forall x \in M \), \( x \) lies in the Voronoi cell of some seed \( G_{ijk} \) where \( x \in B_i \cup B_j \cup B_k \).

**Proof.** Assume the contrary. There must exist a sample \( p_k \) such that \( x \in B_k \) and \( G_{ijk} \notin \partial B_a \). Then, the line segment \( \ell_x = [xG_{ijk}] \), which lies within the Voronoi cell of \( G_{ijk} \), would have to intersect \( B_a \). But, the interior of \( B_a \) near \( p_a \) is decomposed by the Voronoi cells of the seeds on \( \partial B_a \), a contradiction to \( \ell_x \) lying within the Voronoi cell of \( G_{ijk} \).

**Lemma 19.** For any \( x \in M \) and any normal segment \( N_x \) issued from \( x \), the distance to \( G_O \) is either strictly increasing or strictly decreasing along \( g^{-1}(0, 0.96c) \) \( \cap N_x \). The same holds for \( G_I \).

**Proof.** Let \( n_x \) be the outward normal and \( T_x \) be the tangent plane to \( M \) at \( x \). Fix any sample \( p_i \) such that \( x \in B_i \). For all possible locations of a seed \( s \in G_O \cap \partial B_i \), we will show a sufficiently large lower bound on \( (s - s'', n_x) \), where \( s'' \) the projection of \( s \) onto \( T_x \).

Take \( lfs(p_i) = 1 \) and let \( B_i = B(c, 1) \) be the tangent ball to \( M \) at \( p_i \) with \( s \in B_i \). Let \( A \) be the plane containing \( \{p_i, s, x\} \). Assume in the worst case that \( A \parallel T_p \), and \( x \) is as far as possible from \( s \) on \( \partial B_\ast \cap \partial B_x \). As \( d(p_i, x) \leq \delta \), it follows that \( \theta_x = \angle(n_x, n_p) \leq \delta/(1 - \delta) \leq 40\epsilon/19 \). This means that \( T_x \) is confined within a \( (\pi/2 - \theta_x) \)-cocone centered at \( x \). Assume in the worst case that \( T_x \parallel T_p \) is tilted to minimize \( d(s, s') \); see Figure 5b.

Let \( T_x' \) be a translation of \( T_x \) such that \( p_i \in T_x' \) and denote by \( x' \) and \( s' \) the projections of \( x \) and \( s \), respectively, onto \( T_x' \). Observe that \( T_x' \) makes an angle \( \theta_x' \) with \( T_p \). From the isosceles triangle \( \triangle p_x c_x \), we get that \( \theta_x' = 1/2 \angle p_x c_x = \sin^{-1} \epsilon \leq \epsilon \). Now, consider \( \triangle p_x x' \) and let \( \phi = \angle p_x x' \). We have that \( \phi = \theta_x + \theta_x' \leq \epsilon + \delta/(1 - \delta) \leq 3.01 \epsilon \) and \( d(x, x') \leq \delta \sin(\phi) \). On
the other hand, we have that $\angle sp, s' = \psi \geq \theta_s - \theta_x$ and $d(s, s') \geq \delta \sin \psi$, where $\theta_s \geq 1/2 - 5\epsilon$ by Lemma 14. Simplifying we get $\sin(\psi) \geq 0.49 - 6.86\epsilon$. The proof follows by evaluating $d(s, s'') = d(s, s') - d(x, x')$.

![Figure 5](image_url)  
(a) Seed elevation $\theta_s$.  
(b) Bounding seed height above $T_s$.  
(c) Bounding $d(q, M)$.  

**Theorem 20.** For every $p \in M$ with closest point $q \in \hat{M}$, and for every $q \in \hat{M}$ with closest point $p \in M$, we have $\|pq\| < h_t \epsilon^2 \text{lfs}(p)$, where $h_t < 30.52$. For $\epsilon < 1/500$, $h_t \epsilon^2 < 0.0002$. Moreover, the restriction of the mapping $\pi$ to $\hat{M}$ is a homeomorphism and $\hat{M}$ and $M$ are ambient isotopic. Consequently, $\hat{O}$ is ambient isotopic to $O$ as well.

**Proof.** Fix a sample $p_1$ and consider two cocones centered at $p$: a $p$-cocone contains all nearby surface points and a $q$-cocone contains all guide triangles incident at $p_1$. By Theorem 10, all reconstruction facets generated by seeds on $B_1$ are sandwiched in the $q$-cocone. Lemma 17 readily provides a bound on the $q$-cocone angle as $\gamma \leq \eta \delta$. On the other hand, since $d(p_1, p) \leq \delta \text{lfs}(p_1)$, we can bound the $p$-cocone angle as $\theta \leq 2 \sin^{-1} (\delta/2)$ by Lemma 2 in [1]. We utilize a mixed $pq$-cocone with angle $\omega = \gamma/2 + \theta/2$, obtained by gluing the lower half of the $p$-cocone with the upper half of the $q$-cocone.

Fixing $q \in \hat{M}$, consider its closest point $p \in M$; see Figure 5c. By sandwiching, we know that any ray through $q$ intersects at least one guide triangle, in some point $q'$, after passing through $q$. Let us assume the worst case that $q'$ lies on the upper boundary of the $pq$-cocone. Then, $d(p, q) \leq d(p', q') = h = \delta \sin(\omega) \text{lfs}(p_1)$, where $p'$ is the closest point on the lower boundary of the $pq$-cocone point to $q$. We also have that, $d(p, p) \leq \cos(\omega) \delta \text{lfs}(p_1) \leq \delta \text{lfs}(p_1)$, and since $\text{lfs}$ is 1-Lipschitz, $\text{lfs}(p_1) \leq \text{lfs}(p)/(1 - \delta)$. Simplifying, we write $d(p, q) < \delta \omega/(1 - \delta) \cdot \text{lfs}(p) < h_t \epsilon^2 \text{lfs}(p)$.

With $d(p, q) \leq 0.55 \text{lfs}(p)$, Lemma 19 shows that the normal line from any $p \in M$ intersects $\hat{M}$ exactly once close to the surface. It follows that for every point $p \in M$ with closest point $q \in \hat{M}$, we have $d(p, q) \leq d(p, q')$ where $q' \in M$ with $p$ its closest point in $M$. Hence, $d(p, q) \leq h_t \epsilon^2 \text{lfs}(p)$ as well.

Building upon Lemma 19, as a point moves along the normal line at $p$, it is either the case that the distance to $G_O$ is decreasing while the distance to $G_I$ is increasing or the other way around. It follows that these two distances become equal at exactly one point on the Voronoi facet above or below $p$ separating some seed $s^+ \in G_O$ from another seed $s^- \in G_I$. Hence, the restriction of the mapping $\pi$ to $\hat{M}$ is a homeomorphism. With $\hat{M}$ and $M$ homeomorphic and $h_t \epsilon^2 \text{lfs}(p) \leq 0.0002 \text{lfs}(p)$, for $\epsilon \leq 1/500$, it follows that $\hat{M}$ and $M$ are ambient isotopic.
by Theorem 9 of [8]. Finally, as $\hat{\mathcal{M}}$ is the boundary of $\mathcal{O}$ by definition, it follows that $\hat{\mathcal{O}}$ is ambient isotopic to $\mathcal{O}$ as well. ▶

5 Quality Guarantees

See Appendix C for the proof details and additional lemmas.

▶ Corollary 21 (Seed height). If $t_{ijk}$ is a guide triangle with associated seed $s$, then $\angle s_p s'' \geq \frac{1}{2} - \eta'_t \epsilon$, where $s''$ is the projection of $s$ on the plane of $t_{ijk}$ and $\eta'_t \geq 5 + 2 \eta_t > 18.18$, implying $d(s, s'') \geq \hat{h}_s d_lfs(p_i)$ with $\hat{h}_s \geq \frac{1}{2} - \eta'_t \epsilon$.

Proof. Combining Lemma 14 with Lemma 17, we have $\angle s_p s'' \geq \angle s_p s' - \angle a(n_{t_{ijk}}, n_{p_i})$. ▶

▶ Theorem 22 (Triangle Quality). For a guide triangle $t_{ijk}$, edge length ratios are bounded: $\ell_k/\ell_j \leq \kappa_t = \frac{2\delta}{1 - 3\delta}$. angles are bounded: $\sin(\theta_i) \geq 1/(2\rho_f)$ so $\theta_i \in (7.8^\circ, 165^\circ)$. altitude are bounded: the altitude above $e$ is at least $\alpha_t |e|$ where $\alpha_t = 1/(4\rho_f) > 0.067$.

Proof. The sparsity condition ensures a minimum edge length, and the fact that delta balls must overlap provides a maximum edge length. The upper bound on the circumradius provides the angle and altitude bounds. ▶

We now turn to bounding the aspect ratio of the Voronoi cells. We first consider the inradius. An interesting observation is that a guide triangle is contained in the Voronoi cell of its seed, even when one of the guide seeds is covered. Hence the tetrahedron formed by the triangle together with its seed lies inside the cell. Combining the good quality of the guide triangles with the minimum height of the seed above the triangle, we are able to show that this tetrahedron has a lower bound on its inradius.

5.1 Interior Seeds and Bounded Aspect Ratios

We get Voronoi cells with good aspect ratios by using the following octree algorithm to place interior seeds. We obtain provable aspect ratio (outradius to inradius) bounds for all seeds, and a bound on the number of seeds.

▶ Definition 23 (Interior lfs). We extend lfs beyond $\mathcal{M}$, using the point-wise maximal 1-Lipschitz extension [34]: $lfs(x) = \inf_{p \in \mathcal{M}} (lfs(p) + d(x, p))$.

After the VoroCrust guide seeds $\mathcal{G}$ are created, we create additional seeds $\mathcal{I}$ by refining an octree of the domain. A box is refined as long as its radius $r$ (half its diagonal length) is large compared to the lfs: i.e. $r > 0.5 lfs(c)$, where $c$ is the box center. After refinement terminates, we place an interior seed at the center of each empty box, and do nothing with boxes that already contain one or more guide seeds.

The box size will be bounded above and below by the lfs within the box. Further, the box size is naturally balanced, and slowly varying, as in a Lipschitz condition. Any Voronoi vertex is in some box, and every box has at least one seed, hence this provides an upper bound on the distance between a Voronoi vertex and its closest seed. This can be used to get an upper bound on the outradius of a Voronoi cell. Also, interior seeds are at the center of the box, so the box side length provides a lower bound on the inradius of an interior seed. Combining these we are able to show the following theorems.

▶ Theorem 24 (Interior Aspect Bound). Interior seeds aspect ratios $\leq \frac{8\sqrt{3}(1+\delta)}{1-3\delta} < 14.1$. 
Theorem 25 (Guide Aspect Bound). Guide seeds aspect ratios \( \leq \frac{4(1+4\delta)}{(1-3\delta)(1-\delta)} c_o \) < 31.6.

We may also bound the output size, the number of seeds and cells in the Voronoi mesh, in terms of the lfs. We integrate \( 1/lfs^3 \) over the domain. The integral over a single cell is bounded above by a constant, because the bounds on the aspect ratios bounds the variation of the lfs within a cell. Thus the integral in effect sums the cells.

Theorem 26 (Number of Cells). \(|S| \leq c_S \int_O lfs^{-3} \), where seeds \( S = G \cup I \) and \( c_S = c^3_o/c_{ig} < 1.9e+09 \), with \( c_{ig} = \frac{4\pi}{3} (\varrho v \epsilon (1-\delta))^3 \) and \( c_o = 1 + \frac{46(1+\delta)}{(1-3\delta)(1-\delta)} \).

6 Conclusions

Summary. We have shown that VoroCrust can provably generate a conforming Voronoi mesh conforming to any closed curved surface. The mesh comes with quality guarantees. Voronoi cells have bounded aspect ratios. The triangular faces of these cells that make up the reconstruction have bounded angles and edge-length ratios. VoroCrust has an elegance compared to filtering, in that no special-case algorithmic steps are required in the presence of slivers. Instead, the seeds naturally form Steiner points inside a sliver, and the reconstruction is still a surface, without 3D elements. This may provide new insight into the nature of alpha shapes and their duals in the presence of multiple overlapping balls.

Future Work. It would be interesting to extend the algorithm and guarantees to higher dimensions.

On the practical side, our companion papers [32,33] describe VoroCrust sampling and reconstruction algorithms, practical implementations, and other contributions. For example, a coarser sampling often suffices for reconstruction. One simplification would be a sampling algorithm that ensures both guides are uncovered, or both covered. The significance would be that no tetrahedral slivers arise and no Steiner points are introduced; and it becomes easy to show convergence of normals.

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References


Sampling Conditions for Conforming Voronoi Meshing by the VoroCrust Algorithm

A Motivation

The problem of reconstructing an approximating surface given a finite point sample $P$ from an unknown surface $M \subset \mathbb{R}^3$ is inherently ill-posed as there are, in general, several surfaces that could have produced $P$. Therefore, a necessary condition for satisfactory reconstruction is that $P$ is dense enough such that all relevant features of the underlying surface can be detected by examining $P$ only. The $\epsilon$-sampling framework provides a convenient way to describe such necessary conditions.

We start by recalling that the medial axis $\text{medial axis}$ of $M$ is the closure of the set of points $x \in \mathbb{R}^3$ such that the minimum distance from $x$ to $M$ is realized by more than one point on $M$. This allows us to recall the following core concepts.

Definition 27 (Local Feature Size and $\epsilon$-sampling). The local feature size ($\text{lfs}$) of $x \in M$ is its Euclidean distance to the medial axis. A sampling $P$ is an $\epsilon$-sample if $\forall x \in M$, $\|x - p_i\| \leq \epsilon \text{lfs}(x)$ for some sample $p_i \in P$.

This form of sampling was developed for the power crust algorithm [1, 3, 5, 6] and is now standard in the surface reconstruction literature as it provides a convenient theoretical framework to argue that as the sampling density increases, the reconstruction $\hat{M}$ will converge to the original manifold $M$. In particular, many reconstruction proofs require that the input samples are an $\epsilon$-sample for $\epsilon = 0.05$ [19] or some other small value, and in practice a larger $\epsilon$ usually suffices.

Amenta et al. [2, 6] established many useful geometric properties of $\epsilon$-samples that apply equally well to our setting. For small $\epsilon$, the manifold locally resembles a planar surface, i.e., the set of samples enclosed in an $\epsilon$-lfs-ball are within a few degrees of the tangent plane at the center. In other words, the manifold lies within a tight $\text{cocone}$, the complement of a large-angled double cone at the sample [4, 20, 21].

Many reconstruction methods [3, 9, 19, 28, 31], as in VoroCrust, require samples to meet some density conditions that are qualitatively the same as $\epsilon$-sampling; perhaps relaxed at creases. Additionally, some methods require $\text{sparsity}$ with a minimum separation between samples, such as a $k$-tight $\epsilon$-sampling where $\exists k \in (0, 1) : \forall p_i, p_j \in P, \|p_i - p_j\| > k \epsilon \text{lfs}(p_i)$ [27]. Our sufficient conditions also require sparsity. Given a dense sampling, sub-sampling can convert it into a sparse one and estimate the lfs [30]. For some small $\epsilon$ and $\delta > \epsilon$, the union of $\delta$-lfs-balls around samples is homotopy equivalent to the manifold [16]. We require a similar local condition for each sample sphere, and show it is achieved by a sparse sampling.

Homology inference provides an alternative characterization of surface reconstructions from weighted point clouds. As weights are increased from zero, we may track the connectivity the union of balls using persistent homology. This alternative approach has produced several theorems similar to those developed in the $\epsilon$-sampling framework; see, e.g., [12, 14, 15, 18, 35].

Reconstruction Schemes.

Reconstructions based on Delaunay triangulation [13] create primal reconstructions. The power crust [5] is distinguished by its reconstruction being the boundary of some cells of a weighted Voronoi diagram of some constructed seeds. It is closest to our method. The power crust algorithm performs a $\text{primal-dual-dual-primal dance}$: compute the dual of the sample points, select a subset as a new set of primal points, weight them, and dualize them.
to create cells. VoroCrust performs a similar dance, but with key differences in when the weighting is applied, and which Voronoi seeds are selected.

- Power crust uses some far (unweighted) Voronoi vertices of $\text{Vor}(S)$ as seeds, to build a (weighted) power diagram whose faces pass through $S$.
- VoroCrust uses some nearby points on (weighted) power edges of $\text{wVor}(S)$ as seeds, to build a (unweighted) Voronoi diagram whose vertices include $S$.

VoroCrust and power crust may produce different vertices, edges, and triangles for the same samples. Indeed, VoroCrust may produce different triangulations by adjusting sample weights. In VoroCrust, all samples are vertices. In power crust, a sample may appear as a vertex, or interior to an edge or facet. Some variations of power crust use filtering to ensure samples are vertices at the price of the reconstruction no longer being the boundary of power cells [1, 4, 22]. One 2D algorithm uses circles to place Voronoi seeds for segmenting graphics edges around a vertex [24].

Often, the cells of a Voronoi mesh are restricted to the surface [1, 23, 38]. Facets with non-empty intersection are dualized to form the reconstruction facets. Contemporary techniques [25, 36] for Voronoi meshing typically involve creating or moving seeds in a domain, forming their Voronoi cells, then clipping the cells by the boundary surface. However, this can lose some important properties such as convexity and connectedness. Alternatively, as in Lloyd iteration to form a centroidal Voronoi tessellation, one may consider getting a stable reconstruction with well-shaped cells from the fixed points of the restricted Delaunay triangulation [29].

**Practical Advantages.**

(a) Surface reconstruction. Some non-sample Steiner points are circled in red.

(b) Unstructured.

(c) Structured interior.

*Figure 6* Sample VoroCrust reconstructions. The surface is reconstructed by a surface mesh and additional seeds can be added to the interior to make a volumetric mesh of well-shaped polyhedral cells. Surface elements are triangles, except when weighted samples are cocircular resulting in polygonal faces with more sides. Interior volumetric cells are general polyhedra, depending on the regularity of the interior seeds.

Our companion VoroCrust papers [32, 33] demonstrate its practical advantages over other methods, among other contributions. We summarize both the process advantages and output advantages for motivation. The process advantages are simplicity and robustness. The unweighted Voronoi diagram of the seeds produces the output reconstruction. This can be generated by established libraries such as Voro++ [37], without modification or special
Sampling Conditions for Conforming Voronoi Meshing by the VoroCrust Algorithm

Cases. Clipping requires an implementation of robust constructive solid geometry subtraction operations, and filtering requires extra geometric checks.

The second advantage is that of the output. VoroCrust cells are true Voronoi cells, with many desirable properties both in theory and applications, such as convexity and fatness. The volume decomposition has no special cases, unlike filtering which requires bookkeeping of which substructures have been discarded. Unweighted output has advantages over weighted output, e.g., one can determine if a query point is inside the domain by simply finding its nearest Voronoi seed. The output is also advantageous because it is local: seeds lie near the surface. This is in contrast to the power crust family, where seeds lie near the medial axis. This locality provides freedom to place additional seeds interior to the volume, away from the surface, without disturbing the reconstruction. For example, additional interior seeds can create a good-quality volume mesh composed of Voronoi polyhedra. These seeds can be placed randomly, or on a structured grid to produce hexahedral cells; see Figure 6.

B Sandwich Analysis

The facet separating the two uncovered seeds on an intersection circle is sandwiched, lying between the two corresponding guide facets; it also contains the edge dual to the circle. There are four facets separating a lower and upper guide, the red ones in Figure 4a. These meet at a Voronoi vertex \( n \) interior to the sliver, sandwiched between the upper and lower triangle pairs. This vertex \( n \) is a Steiner point of the surface mesh, a vertex that was not a sample. Note that in general \( n \) is not the weighted circumcenter of the sliver, but it does tend to lie close to it when weights are about the same. This “sandwiching” also holds in more general cases of multiple adjoining slivers; see Lemma 30.

We shall see that each sample is still surrounded by a fan of facets, meaning the set of facets containing the sample is a topological disk with the sample in its interior.

We first establish some properties about the angular orientation of the fan facets around \( p_1 \), namely that they lie sandwiched between upper and lower guide triangles; their extent is not considered until near the end, at Lemma 30. For orientation, it suffices to consider only the seeds on \( S_1 \), because these are the seeds whose cells contain \( p_1 \), and hence define the fan facet orientations. By excluding other seeds, the fan facets are all triangles extending radially from \( p_1 \) to infinity. This allows us to make the relevant orientation arguments on the surface of \( S_1 \), using the spherical arcs formed by the intersections of extended facets with \( S_1 \). These arcs are great circle arcs between the two points where the two edges bounding the facet intersect \( S_1 \) as they extend from \( p_1 \) to infinity. Later, in Lemma 30, we will clip fan-facets and reintroduce the other seeds.

Pick any facet \( t \) radially extending from \( p_1 \). Now \( t \) is defined by the intersection of Voronoi cells for some upper and lower seeds, which we denote by \( G_{123}^\uparrow \) and \( G_{145}^\downarrow \) to focus on the case when these seeds are not a guide pair, but come from the intersection of \( S_1 \) with different sets of spheres. Let \( m \) be any of the points of intersection of \( t \) with \( S_1 \), called an extended fan-facet point. We will show that \( m \) lies sandwiched between guide triangles, in the medial wedge region. Since \( m \) and \( t \) were arbitrary, this shows the entirety of all reconstructed triangles are sandwiched. Most of the proofs in this section are technical arguments about paths on the sphere \( S_1 \) crossing a cap or a crown, as illustrated in Figure 7.

\[ \text{Lemma 28 (Path } m \text{ to } G^\downarrow \text{ misses } K^\uparrow) \]. For any facet having \( p_1 \) as a vertex, the shorter great circle arc \( mG^\downarrow \) on \( S_1 \), from any extended fan-facet point \( m \) to its lower seed point \( G^\downarrow \), does not pass through \( K^\uparrow \).
Proof. WLOG assume \( i = 1 \) and fix one such facet. Denote the upper and lower seeds associated with the extended facet by \( G_{123}^1 \) and \( G_{145}^1 \). Since a facet point lies on the intersection between the Voronoi cells of the two associated seeds, \( m \) is the center of a sphere \( S_m \) of radius \( \| mG_{145}^1 \| \) touching \( G_{123}^1 \) and \( G_{145}^1 \), and having an interior empty of seeds. The intersection of the empty sphere \( S_m \) with \( S_1 \) is the circle \( C_m \), which we think of as a disk embedded on the surface of \( S_1 \) and also has an interior empty of seeds. The proof follows by examining great circle arcs on \( S_1 \) restricted within \( S_m \). Recall the upper rim \( \partial K_1^0 \) forms a closed path, as \( K_1^0 \) is a disk, and \( \partial K_1^1 \) is disjoint from the lower rim \( \partial K_1^0 \), as \( K_1^1 \) and \( K_1^0 \) are separated by a medial band. Being a lower seed, \( G_{145}^1 \) lies on \( \partial K_1^0 \).

Now suppose for contradiction that \( mG_{145}^1 \) goes through \( K_1^0 \) and let \( x \in mG_{145}^1 \subset C_m \) be the last point where \( mG_{145}^1 \) crosses \( \partial K_1^0 \) in the direction from the interior of \( K_1^0 \) to its exterior. We have that \( x \) lies on an uncovered arc of a circle of intersection \( C_{16} \) between \( S_1 \) and some \( S_6 \) with seeds \( G_{167}^1 \) and \( G_{168}^1 \) as endpoints, i.e., \( x \in G_{167}^1G_{168}^1 \subset K_1^0 \). Figure 7a depicts such a hypothetical path \( mG_{145}^1 \). Since \( C_m \) is empty, \( G_{167}^1 \) and \( G_{168}^1 \) both lie outside \( C_m \). In addition, \( G_{145}^1 \) cannot lie inside \( C_{16} \), as \( G_{145}^1 \) is an uncovered seed and \( C_{16} \) bounds the region on \( S_1 \) which is covered by \( S_6 \); this impossible configuration is illustrated by the dashed circle in Figure 7a. It would follow that \( mG_{145}^1 \) must cross \( G_{167}^1G_{168}^1 \) a second time to reach \( G_{145}^1 \), a contradiction to \( mG_{145}^1 \) intersecting \( \partial K_1^0 \). (Special cases include \( K_1^0 \) and \( K_1^1 \) meeting at a point, or \( x \) being a point of tangency between \( C_{16} \) and \( mG_{145}^1 \).

A reconstructed fan facet is sandwiched at a sample if the intersection of its radial extension with the surface sphere lies in the medial wedge.

Lemma 29 (Sandwich for Fan Facets). All reconstruction facets containing a sample as a vertex are sandwiched between guide triangles.

Proof. Again, the proof follows by analyzing spherical arcs on \( S_1 \) which arise from intersecting \( S_1 \) with extended reconstruction facets having \( p_i \) as a vertex.

On the surface of \( S_1 \), we partition the area above the crown of upper guide triangles, and show that no point \( m \) of an extended reconstruction facet can lie in any of these partitions. The argument for lower triangles is analogous. See Figure 7b. By Lemma 28, \( m \) cannot lie in the upper cap. Hence, we focus on the regions below the upper rim.

On the surface of \( S_1 \), consider the arcs composing the upper rim, WLOG arcs \( G_{123}^1G_{145}^1 \) of the circle \( C_{12} \), with center \( e_{12} \), where some \( S_2 \) intersects \( S_1 \) and \( e_{12} \) is taken to mean the intersection of the ray \( \overrightarrow{p_1p_2} \) with \( S_1 \). Each consecutive pair of such circles, which all lie on the surface of \( S_1 \), intersect at an uncovered upper guide point \( G_{1} \) and a lower guide point \( g_{1} \), which can be covered or uncovered. We partition \( C_{12} \) by great circle arcs \( G_{123}^1G_{124}^1 \) and their angle bisector through \( e_{12} \). WLOG consider the partition containing \( G_{123}^1 \). We further partition into \( A_1 \), above \( e_{12}G_{123}^1 \) \( A_2 \), above upper guide facet \( \triangle_{123} \); and \( B_1 \), below \( \triangle_{123} \). Suppose there is some point \( m \) on an extended reconstruction facet whose two closest and equidistant seeds are \( G_{123}^1 \) and some lower seed which we denote by \( G_{156}^1 \). We show that \( m \) cannot lie above \( \triangle_{123} \), neither in \( A_1 \) nor \( A_2 \).

Suppose \( m \in A_1 \) and consider the circle \( C_m \) on \( S_1 \), with center \( m \) passing through \( G_{123}^1 \). As no seed is closer to \( m \) than \( G_{123}^1 \), \( C_m \) is empty of other seeds and cannot contain \( G_{124}^1 \) in its interior. Hence, the only portion of \( C_m \), outside \( C_{12} \) lies above \( G_{123}^1G_{124}^1 \) (see Figure 7c.
Lemma 28: arcs from reconstruction $m$ to $G^i$ cannot cross $K^T$.

Lemma 29: partitioning $m$’s hypothetical locations: upper cap, $A_1$, $A_2$, and $B$.

Lemma 29: $m \notin A_1$, otherwise $G^i_{156}$ would lie in or beyond the upper cap.

Lemma 29: $m \notin A_2$, otherwise if $C_{13}$ is shorter than $e_{12}m$ and $G^i_{156} = g^i_{123}$.

Lemma 29: $m \notin A_2$, otherwise if $e_{12}m$ is longer, $mG^i_{156}$ must cross $K^T$.

Figure 7 Edge and fan-facet sandwich lemmas.

It follows that $mG^i_{156}$ must cross into the upper cap, a contradiction to Lemma 28 and we conclude $m \notin A_1$.

Suppose $m \in A_2$ and consider the circle $C_{13}$ with center $e_{13}$ passing through $G^i_{123}$. We have two subcases: either $e_{12}m$ is shorter than $e_{12}e_{13}$ or it is longer. Suppose $e_{12}m$ is shorter than $e_{12}e_{13}$; see Figure 7d. In this case, $C_m$ lies entirely in the interior of $C_{12} \cup C_{13}$, except at $G^i_{123}$ and perhaps $g^i_{123}$. It would follow that the only possible position for $G^i_{156}$ is $g^i_{123}$, which is a contradiction since $G^i_{156}$ is uncovered while $g^i_{123}$ is covered. In the second case, $e_{12}m$ is longer than $e_{12}e_{13}$, with $m$ inside $C_{13}$; see Figure 7e. In this configuration, some arc $G^i_{123}G^i_{137}$ of $C_{13}$ towards $g^i_{123}$ is in the upper cap, where again $G^i_{137}$ cannot be inside $C_m$.

As $G^i_{156}$ cannot coincide with $g^i_{123}$, $mG^i_{156}$ strictly crosses $G^i_{123}G^i_{137}$ into $K^T_1$, a contradiction to Lemma 28. As both cases cannot be true, we conclude $m \notin A_2$.

Lemma 30 (Sandwich for all Facets). Every reconstruction vertex is sandwiched between upper and lower guide facets.

Proof. WLOG, every upper seed $G^i_{123}$ has Voronoi facets containing each of the sample vertices of the associated guide triangle $\Delta_{123}$, but they need not be in a common facet.
Lemma 33. We make the following observations about the octree from Section 5.1.

C.1 Bounded Aspect Ratio Proofs

WLOG, consider a reconstructed facet $f_1$ associated with the seed $G_{123}^t$ and containing $p_1$. As the Voronoi cell $\text{Vor}(G_{123}^t)$ is convex, it cannot extend below $f_1$. By Lemma 29, all vertices of $f_1$ are sandwiched above the lower guide triangles. Consequently, $\text{Vor}(G_{123}^t)$ cannot extend below any of the lower guide triangles. In particular, any Steiner vertex $v$ from $\text{Vor}(G_{123}^t)$ appearing in the reconstruction is above these lower guide facets. We now sandwich $v$ from above, by repeating the argument for all lower seeds $G_{1}^t$ where $v \in \text{Vor}(G_{1}^t)$.

C Quality Guarantee Proofs

Here we provide the formal arguments for the proofs in Section 5.

Theorem 31 (Triangle Quality). For a guide triangle $t_{ijk}$,
- edge length ratios are bounded: $\ell_k/\ell_j \leq \kappa = \frac{26}{1+3 \cdot 165^2}$.
- angles are bounded: $\sin(\theta_i) \geq 1/(2\rho_{ij})$ so $\theta_i \in (7.8^\circ, 165^\circ)$.
- altitude are bounded: the altitude above $e$ is at least $\alpha_e |e|$, where $\alpha_e = 1/4\rho_{ij} > 0.067$.

Proof. Denote by $\ell_i$ and $\theta_i$ the length of the triangle edge opposite to $p_i$ and the angle at vertex $p_i$, respectively. Observation 15 implies $\ell_k \leq \kappa \ell_f(p_i)$ and the sparsity condition guarantees that $\ell_j \geq \kappa \ell_f(p_j)$, hence $\ell_i/\ell_k \leq \kappa$ for any pair of edges.

Let $R_{ijk}$ denote $t_{ijk}$’s circumradius. By the Central Angle Theorem, $\sin(\theta_i) = \ell_i/(2R_{ijk})$, and we also have $R_{ijk} \leq \rho_{ij} \ell_i$ from Lemma 16. Hence $\sin(\theta_i) \geq 1/(2\rho_{ij})$.

For the worst case altitude, let the edge under consideration be the longest, $e = \ell_k$, and the second longest edge $\ell_j$, so $\ell_j \geq \ell_k/2$. The altitude is then $\sin(\theta_i)\ell_j \geq \ell_k/(4\rho_{ij})$.

Lemma 32. For a seed $G_{ijk}$, the inradius of its Voronoi cell is at least $r \geq \rho_v \ell_f(p_i)$ with $\rho_v = \hat{h}_s/\sqrt{\frac{3}{4\rho_{ij}}} > 0.26$.

Proof. Fix a seed $G_{ijk}$ and observe that $\{p_i, p_j, p_k\}$ belong to its Voronoi cell. By the convexity of the cell, it follows that the tetrahedron $T = p_ip_jp_kG_{ijk}$ is contained inside it. We establish a lower bound on the cell’s inradius by bounding the inradius of $T$. Let $f_i$ denote the facet of $T$ opposite to $p_i$ and $f_0$ denote $t_{ijk}$. Let $A_i$ be the area of $f_i$.

Observe that the incenter $C_T$ divides $T$ into four smaller tetrahedra, one for each facet of $T$, where the distance from $C_T$ to the plane of each facet is equal to the inradius $r$. This allows us to express the volume of $T$ as $V = \sum_{i=0}^{3} rA_i/3$. Hence, we have that $r = 3V/\sum A_i$. We may also express $V$ as $H A_0/3$, where $H$ is the distance from $G_{ijk}$ to the plane of $t_{ijk}$. Substituting for $V$ and factoring out $A_0$, we get that $r = H/(1 + \sum_{i>0} A_i/A_0)$.

In order to bound $A_i/A_0$, consider the edge $e_i = [p_jp_k]$ common to $f_i$ and $t_{ijk}$ and let $\alpha$s and $\alpha_p$ be the altitudes of $e_i$ in $f_i$ and $t_{ijk}$, respectively. It follows that $A_i/A_0 = \alpha_s/\alpha_p$. Note $\alpha_s$ is less than the length of the longest edge of $f_i$. Hence, assuming WLOG that $\ell_f(p_j) \geq \ell_f(p_k)$, we get that $\alpha_s \leq \ell_f(p_j)$. On the other hand, the sparsity condition guarantees $\rho_{ij} \geq \rho_{jk} \ell_f(p_j)$, allowing us to rewrite $\alpha_s \leq \frac{1}{2} \rho_{ij} \ell_f(p_j)$. From Lemma 27, we have that $\alpha_p \geq \ell_f(p_k)/(4\rho_{ij})$. It follows that $A_i/A_0 \leq \frac{1}{2} \rho_{ij} \ell_f(p_i)$. The proof follows by invoking Corollary 21 to bound $H \geq \hat{h}_s \rho_{ij} \ell_f(p_i)$.

C.1 Bounded Aspect Ratio Proofs

We make the following observations about the octree from Section 5.1.

Lemma 33. For every leaf box, $r \in [\frac{1}{4+\delta}, 1] \ell_f(s)$. 

Proof. By definition the leaf box was not split, so \( r \leq \delta \text{lfs}(c) \). But its parent was split, so \( 2r > \delta \text{lfs}(k) \), where \( k \) is the parent’s center and a leaf box corner. The 1-Lipschitz condition on \( \text{lfs} \) then gives \( \text{lfs}(c) \leq \text{lfs}(k) + r < r(1 + 2/\delta) \).

\[ \text{Lemma 34. For any point } p \text{ in a leaf box, } r \in \left[ \frac{1}{2 \sqrt{3}}, \frac{1}{\sqrt{3}} \right] \delta \text{lfs}(p). \]

Proof. Apply Lemma 33, then observe point \( p \) is at most \( r \) from its box center, so \( \text{lfs}(p) \) is bounded above and below in terms of \( \text{lfs}(c) \).

\[ \text{Lemma 35 (Balanced Octree). For boxes } \{1, 2\} \text{ sharing a corner, } 2r_2 \geq r_1 \geq r_2/2, \text{ for } \delta < 1/3. \]

Proof. WLOG let \( r_2 < r_1 \). From Lemma 33 and \( \text{lfs} \) is 1-Lipschitz, \( r_2 \geq \frac{\delta}{2 + \sqrt{3}} \text{lfs}(c_2) \geq \frac{\delta}{2 + \sqrt{3}}(r_1/\delta - r_1 - r_2) \). Thus \( r_2 \geq \frac{\delta}{2 + \sqrt{3}} \). But box sizes are powers of 2, and \( \delta < 1/3 \) implies \( r_2 > r_1/4 \).

These may be used to bound the outradius of Voronoi cells, which is then combined with the inradius bound to provide an aspect ratio bound.

\[ \text{Lemma 36 (Outradius). Interior and guide seeds have cell outradius at most }\frac{4\sqrt{(1+\delta)^{1+\delta}}}{1-\delta}r_s. \]

Proof. Consider the box containing any Voronoi vertex \( v \), and observe its distance to one of its seeds \( s \) is at most \( 2r_v \), because all boxes contain some seed. Applying Lemma 34 to the box of \( v \), \( d(v, s) \leq 2r_v \leq \frac{2\sqrt{3}}{3} \text{lfs}(v) \). But \( \text{lfs}(s) \geq \text{lfs}(v) - d(v, s) \geq \frac{1+\delta}{2+\sqrt{3}}d(v, s) \). Applying Lemma 34 to the box of \( s \), \( d(v, s) \leq \frac{\sqrt{3}}{2-\delta} \text{lfs}(s) \leq \frac{4\sqrt{(1+\delta)^{1+\delta}}}{1-\delta}r_s. \)

\[ \text{Theorem 37 (Interior Aspect Bound). Interior seeds aspect ratios } \leq \frac{8\sqrt{3}(1+\delta)}{1-\delta} < 14.1. \]

Proof. Since no other seed lies in a box, the cell inradius for an interior seed is at least \( r/(2\sqrt{3}) \), half distance from the box center to a closest point on its boundary. Then apply Lemma 36.

\[ \text{Theorem 38 (Guide Aspect Bound). Guide seeds aspect ratios } \leq \frac{4\sqrt{(1+\delta)^{1+\delta}}}{(1-\delta)^2}c_{\text{cav}} < 31.6. \]

Proof. Lemma 32 gives a lower bound on the inradius in terms of \( \text{lfs}(p_i) \). We use the Lipschitz condition to express it in terms of \( \text{lfs}(G_{ijk}) \), namely \( g_v \epsilon (1 - \delta) \text{lfs}(G_{ijk}) \). Then apply Lemma 36, using Lemma 34 to substitute \( G_{ijk} \) for \( r_s \); that is, \( r_s \leq \frac{\delta}{1-\delta} \text{lfs}(G_{ijk}) \).

\[ \text{Theorem 39 (Number of Cells). } |S| \leq c_S \int_{V} \text{lfs}^{-3}, \text{ where seeds } S = G \cup I \text{ and } c_S = c_\text{cav}^3 / c_{\text{cav}} < 1.9e+09, \text{ with } c_{\text{cav}} = \frac{4\sqrt{3}}{3} (g_v \epsilon (1 - \delta))^3 \text{ and } c_0 = 1 + \frac{4\sqrt{(1+\delta)^{1+\delta}}}{(1-\delta)^2}. \]

Proof. Since Voronoi cells partition the domain, \( \int_{V} \text{lfs}^{-3} = \sum_{s \in S} \int_{\text{Vor}(s)} \text{lfs}^{-3} \). Bounded outradii and inradii will bound each integral by constant \( c_S \), as follows.

From Lemma 36, \( \text{lfs} \) is 1-Lipschitz, and Lemma 34, for any \( x \in \text{Vor}(s) \), we have \( \text{lfs}(x) \leq c_o \text{lfs}(s) \), where \( c_o = 1 + \frac{4\sqrt{(1+\delta)^{1+\delta}}}{(1-\delta)^2} \). Thus \( \int_{\text{Vor}(s)} \text{lfs}^{-3} \geq c_o^{-3} \int_{\text{Vor}(s)} \text{lfs}^{-3} \text{vol}(\text{Vor}(s)). \)

The inradius provides a lower bound on the volume. For interior seeds, Lemma 34 yields \( r_i \geq \frac{\delta \text{lfs}(s)}{2\sqrt{3}(2+\delta)^{1/2}} \), so \( \text{vol}(\text{Vor}(s) \in I)) \geq c_i \text{lfs}^3(s), \) where \( c_i = \frac{4\sqrt{3}}{3} \left( \frac{\delta}{2\sqrt{3}(2+\delta)^{1/2}} \right)^3 \). Thus \( \int_{\text{Vor}(s)} \text{lfs}^{-3} \geq c_i / c_0^3 \) for interior seeds.

For guide seeds, Lemma 32 gives \( r_i \geq g_v \epsilon \text{lfs}(p_i) \), but \( \text{lfs}(s) \geq (1 - \delta) \text{lfs}(p_i) \), thus \( \int_{\text{Vor}(s)} \text{lfs}^{-3} \geq c_{\text{cav}}^3 / c_{\text{cav}}^3 \) for guide seeds, where \( c_{\text{cav}} = \frac{4\sqrt{3}}{3} (g_v \epsilon (1 - \delta))^3 \).

Observe \( c_i \geq c_o \). Hence the integral over a single cell is at most \( c_{\text{cav}}^3 / c_{\text{cav}}^3 \).